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The Josephson effect in superfluid helium and general relativity†

Jeeva Anandan‡

Max-Planck-Institut für Physik und Astrophysik, Werner Heisenberg-Institut für Physik, D-8000 München 40, Federal Republic of Germany‡ and Department of Mathematics, University of California, Berkeley, CA94720, USA

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Abstract. A theory of the DC Josephson effect in superfluid helium is presented. The particular case of a toroidal tube with a Josephson junction, containing superfluid helium, is considered. The phase shift, the coupling mass and energy, when this superfluid interferometer has an angular velocity, are obtained. The possibility of detecting the earth's rotation and the general relativistic Lense–Thirring field due to the rotating earth, using this interferometer, is considered. Thermal fluctuations in the superfluid and the consequent limitation on the sensitivity of the interferometer are also studied. Comparison is made with the analogous case of a superconducting ring with a Josephson junction.

1. Introduction

The attractive possibility of detecting general relativistic gravitational effects on superconductors was first suggested by De Witt (1966) and Papini (1967). This idea was recently revived by Widom *et al* (1981), Anandan (1981a, b, c) and Chiao (1981, 1982) and was extended to superfluid helium by the last two authors. Superfluid helium, being neutral, has the advantage over superconductors that no electromagnetic effects need to be taken into account. The purpose of the present paper is to make a careful study of the Josephson effect in superfluid helium and investigate the influence of rotation and gravitation on a Josephson interferometer consisting of a toroidal tube containing superfluid helium interrupted by a Josephson junction. By superfluid helium, here we mean the superfluid of either helium-four atoms or the Cooper pairs of helium-three atoms.

Since there is much controversy regarding the experimental detection of the Josephson effect in superfluid helium (Anderson and Richards 1975, Gamota 1974) it appears necessary to give a derivation of the Josephson effect in superfluid helium in order to convince ourselves that such an effect should be present on general quantum mechanical grounds. We therefore give a simple derivation of the DC Josephson effect for superfluid helium, in § 2, using the Gross–Pitaevsky equation. Since the latter equation is similar to the Ginzburg–Landau equation for superconductors (Ginzburg and Landau 1950) this derivation is also easily extended to superconductors. The reason why this effect has been observed clearly in superconductors but not in superfluid helium-four may be partly because the coherence length in superfluid helium-four

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‡ Present address.

(~10 Å) is much smaller than the coherence length for superconductors (~1000 Å), which makes it so much more difficult to construct a Josephson junction for superfluid helium-four. For superfluid helium-three, on the other hand, the coherence length is ~500 Å; but, in this case, much lower temperatures are needed to produce the superfluid. In this paper, the stated effects on superfluid helium are computed on the assumption that these technical problems can be overcome and a suitable Josephson junction for superfluid helium can be constructed.

In § 3, the Josephson effect is considered for the special case of a toroidal tube, containing superfluid helium, which has an angular velocity along its axis of symmetry. This treatment, which takes into account the relative motion between the superfluid helium and the tube due to the Josephson effect in determining the phase difference across the Josephson junction, differs from some previous treatments (Anandan 1981a, b, Chiao 1981, 1982), but is consistent with Anandan (1981c). The coupling mass and energy for the interaction between superfluid helium and the apparatus is obtained in § 4.

In § 5, the effect of the general relativistic Lense–Thirring field is treated quite simply by noting that this field results in a local precession of inertial frames. This enables the results of § 3 to be immediately applied to this general relativistic situation. A comparison is made with the superconducting analogue of a superconducting ring with a Josephson junction in § 6 and it is shown how the phase shift for both cases can be obtained as special cases of the same basic equation. The effect of thermal fluctuations is considered on § 7 and the limitation on the sensitivity due to thermal fluctuations is determined. We conclude by making a suggestion for future experiments.

2. The DC Josephson effect

We shall make the usual assumption that the superfluid helium is described by an order parameter $\psi(\boldsymbol{r}, t)$ which is a complex function analogous to the usual Schrödinger wavefunction. But it will be assumed to satisfy the Gross–Pitaevsky equation (Gross 1961, Pitaevsky 1961)

$$\mathrm{i}\hbar\partial\psi/\partial t = -(\hbar^2/2m)\nabla^2\psi + V\psi + g|\psi|^2\psi \tag{2.1}$$

where the nonlinear term represents collective interactions of the superfluid, g is a constant and V is a real function representing the 'potential energy' due to the other interactions. It follows from (2.1) that

$$(\partial/\partial t)(\psi^*\psi) + \mathrm{div}[(\hbar/2\mathrm{i}m)(\psi^*\nabla\psi - \psi\nabla\psi^*)] = 0 \tag{2.2}$$

which is formally the same as the continuity equation obtained from Schrödinger's equation. Also in a stationary situation, we can write $\psi(\boldsymbol{r}, t) = \psi_0(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}Et/\hbar}$, where E is a constant, and ψ_0 then satisfies

$$-(\hbar^2/2m)\nabla^2\psi_0(\boldsymbol{r}) + V\psi_0(\boldsymbol{r}) + g|\psi_0|^2\psi_0(\boldsymbol{r}) = E\psi_0(\boldsymbol{r}). \tag{2.3}$$

Writing $\psi_0 = \alpha\,\mathrm{e}^{\mathrm{i}\phi}$, where the 'amplitude' α and the 'phase' ϕ are real functions, the current density

$$\boldsymbol{j} \equiv (\hbar/2\mathrm{i}m)(\psi^*\nabla\psi - \psi\nabla\psi^*) = (\hbar/m)\alpha^2\nabla\phi. \tag{2.4}$$

According to (2.2), $\boldsymbol{j}$ satisfies $\mathrm{div}\,\boldsymbol{j} = 0$ in a stationary situation. On defining the superfluid velocity $\boldsymbol{v}_s \equiv (\hbar/m)\nabla\phi$, since ψ_0 is single valued,

$$(m/\hbar)\oint_\gamma \boldsymbol{v}_s \cdot d\boldsymbol{r} = 2\pi n \tag{2.5}$$

n being an integer, for any closed curve γ which is in the interior of the superfluid.

Consider now two regions containing the superfluid, separated by a Josephson junction, by which we mean a weak link that allows the wavefunction on either side to tunnel through it. We shall assume that a coordinate system can be chosen such that the junction is at rest in the region $-a \leq x \leq a$, V has a constant positive value U inside the junction and $V = 0$ for $|x| > a$. If $U > E$ then ψ in the region $|x| < a$ can be intuitively regarded as a superposition of two decaying wavefunctions from either side. We can therefore obtain an approximate solution of (2.3), in this case, such that

$$\psi_0 = (\alpha_0/\sqrt{2})\,[e^{i\phi_1}\,e^{-(x+a)/\xi} + e^{i\phi_2}\,e^{(x-a)/\xi}], \qquad -a \leq x \leq a, \tag{2.6}$$

where ϕ_1 and ϕ_2 are the values of the function ϕ at $x = -a$ and $x = a$ and α_0, ξ are constants.

It follows from (2.6) that the current density in the junction is

$$\boldsymbol{j} = (\hbar\alpha_0^2\,e^{-2a/\xi}/m\xi)\sin(\phi_2 - \phi_1)\boldsymbol{n} \tag{2.7}$$

where $\boldsymbol{n}$ is a unit vector along the positive x-axis. The current given by (2.7) is analogous to the Josephson current (Josephson 1962) for superconductors and the derivation given here is similar to that of de Bruyn Ouboter (1973) for the superconducting Josephson current. Outside the junction ($|x| > a$), $\boldsymbol{j} = \alpha^2 v_s$ from (2.4). But since $\operatorname{div}\boldsymbol{j} = 0$ everywhere, j has the same value inside and outside the junction. Therefore, setting $\Delta\phi = \phi_2 - \phi_1$, using (2.4) and (2.7)

$$v_s = (\hbar\alpha_0^2\,e^{-2a/\xi}/m\alpha^2\xi)\sin\Delta\phi. \tag{2.8}$$

Also (2.7) gives not only the magnitude of the Josephson current but also its direction. For instance, when $0 < \phi_2 - \phi_1 < \pi$, $\boldsymbol{j}$ is in the direction of increasing phase. This will become important in § 3 when (2.7) or (2.8) will be applied to a specific case.

3. The Josephson effect for a toroidal tube containing superfluid helium

Consider a hollow torodial tube of length L, containing superfluid helium and a Josephson junction of length l inside the tube (figure 1). The thickness of the torus

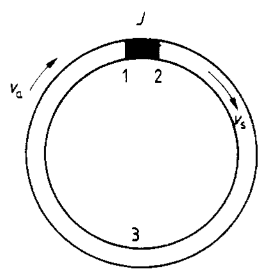

Figure 1. Schematic diagram of a toroidal tube containing superfluid helium and a Josephson junction J. As the tube rotates with tangential speed v_a, the superfluid flows with speed v_s inside the tube satisfying the equation given in the text.

is small compared with the radius. Suppose that the torus rotates about an axis perpendicular to its plane passing through the centre so that the speed at each point on the tube is v_a relative to an inertial frame K. It is clear from the derivation of (2.7) or (2.8) that they are valid relative to the junction. In the present case, the velocity of the superfluid relative to the junction is $v_s - v_a$ where v_s is the superfluid velocity relative to K, outside the junction. Therefore (2.8) must be replaced, in the present case, by

$$v_s - v_a = v_0 \sin(\Delta\phi - mv_a l/\hbar). \tag{3.1}$$

Also it follows from (2.5) that

$$\Delta\phi + (m/\hbar)\int_C \boldsymbol{v}_s \cdot \mathrm{d}\boldsymbol{l} = 2\pi n \tag{3.2}$$

where C is a curve inside the superfluid connecting the two ends of the Josephson junction without passing through the junction and n is an integer.

It follows from the continuity equation $\operatorname{div} \boldsymbol{j} = 0$ that the current $\int_\Sigma \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S}$ (Σ is any cross section of the tube) is constant along the tube. Since the tube is assumed to have uniform cross section and α may be assumed constant in the tube outside the junction, v_s is also constant along C. Hence, from (3.1) and (3.2),

$$v_a - v_s = v_0 \sin(v_s/v_q + mv_a l/\hbar) \tag{3.3}$$

where $v_q = \hbar/m(L-l)$. But it should be noticed that the superfluid velocity inside the junction need not be the same as the v_s outside the junction, because α inside the junction is, in general, different from the α outside the junction. From (3.3),

$$\mathrm{d}v_s/\mathrm{d}v_a = [1 + (v_0/v_q)\cos(v_s/v_q + mv_a l/\hbar)]^{-1}. \tag{3.4}$$

Hence if $v_0 < v_q$, then $\mathrm{d}v_s/\mathrm{d}v_a$ is always well defined and therefore for each v_a, there is a unique solution for v_s in (3.3). Figure 2(a) illustrates v_s as a function of v_a in this case. But if $v_0 > v_q$ then v_s is no longer a single-valued function of v_a and this is illustrated in figure 2(b). In the latter case, hysteresis occurs. It will be assumed from now onwards that $v_0 \leqslant v_q$ so that there is no hysteresis. We shall also assume, for simplicity, that $l \ll L$ so that $v_q \simeq \hbar/mL$ and (3.3) and (3.4) can be approximated by (Anandan 1981c)

$$v_a - v_s = v_0 \sin(v_s/v_q) \tag{3.3$'$}$$

and

$$\mathrm{d}v_s/\mathrm{d}v_a = [1 + (v_0/v_q)\cos(v_s/v_q)]^{-1} \tag{3.4$'$}$$

provided $v_a l/v_q L \ll 2\pi$.

It would be instructive to compute the phase shift due to a *small* change δv_a of the speed of rotation of the apparatus. This will be useful in § 5 where such a small perturbation due to a general relativistic effect arising from the earth's rotation will be considered. The corresponding change in v_s is then $\delta v_s = (\mathrm{d}v_s/\mathrm{d}v_a)\,\delta v_a$, which on using (3.4′), gives the phase shift (the change in $-\Delta\phi$) to be

$$\delta\phi \equiv \frac{\delta v_s}{v_q} = \frac{\delta v_a}{v_q[1 + (v_0/v_q)\cos(v_s/v_q)]}. \tag{3.5}$$

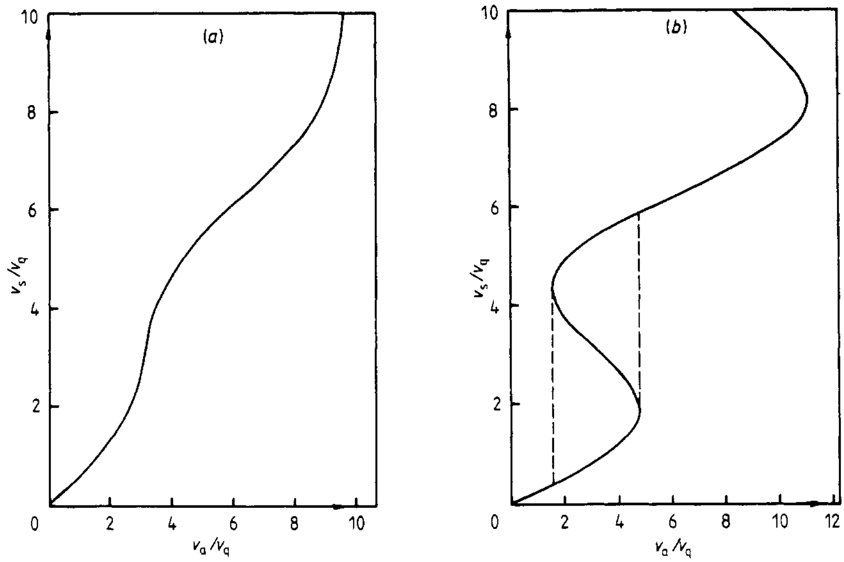

Figure 2. The graph of the superfluid velocity v_s against the apparatus velocity v_a for $v_0/v_q=\frac{3}{4}$ (*a*) and $v_0/v_q=3$ (*b*). For $v_0<v_q$, v_a and v_s are well defined functions of each other. But for $v_0>v_q$, v_s can have several values for a given v_a. In the latter case, hysteresis can occur as indicated by the dotted lines. Both graphs are plotted in the limit $l \ll L$.

We can also write

$$\delta\phi=\frac{m}{\hbar}\oint\frac{\delta v_a\cdot d\boldsymbol{r}}{1+(v_0/v_q)\cos(v_s/v_q)}. \tag{3.6}$$

It then appears that when v_0 is nearly equal to v_q, by appropriately choosing v_a so that $1+(v_0/v_q)\cos(v_s/v_q)$ is small, $\delta\phi$ can be made large.

If the apparatus is at rest with respect to the earth then, in (3.5), $\delta v_a=\Omega_n R$, where Ω_n is the component of the earth's angular velocity normal to the plane of the interferometer and R is the radius of the torus. Hence if the interferometer is not in a horizontal plane and it is turned about a vertical axis, then Ω_n and hence $\delta\phi$ would vary in general. The effect of earth's rotation on this interferometer can then, in principle, be detected by measuring the corresponding change in the Josephson current for the different orientations of the apparatus.

We conclude this section by noting that the treatment of Anandan (1981b) is a special case of the above treatment corresponding to the velocity of the superfluid relative to the apparatus outside the Josephson junction being small compared with v_q. Also, the above treatment and results differ very much from Chiao (1981, 1982).

4. The coupling mass and energy

It was suggested by Chiao (1982) that the superfluid helium Josephson current may be experimentally determined by suspending the apparatus by means of a torsional oscillator and measuring the change in the resonance frequency arising from the recoil due to the flow of the Josephson current. Suppose T is the torque in the suspending string, which is assumed to be along the axis of symmetry of the apparatus. Then, if

L_s and L_a are respectively the angular momenta of the superfluid and the apparatus about this axis

$$T = \mathrm{d}L_s/\mathrm{d}t + dL_a/\mathrm{d}t \tag{4.1}$$

where t denotes the time. Now $L_s = RM_s v_s$ and $L_a = I\Omega_a$, where M_s is the mass of the superfluid, R is the radius of the torus and I, Ω_a are respectively the moment of inertia and angular velocity of the apparatus. It follows from (3.3′) that

$$\frac{\mathrm{d}v_s}{\mathrm{d}t} = \frac{R\,\mathrm{d}\Omega_a/\mathrm{d}t}{1 + (v_0/v_q)\cos(v_s/v_q)}. \tag{4.2}$$

Therefore from (4.1) and (4.2), $T = (I + M_c R^2)\mathrm{d}\Omega_a/\mathrm{d}t$ where

$$M_c = \frac{M_s}{1 + (v_0/v_q)\cos(v_s/v_q)}. \tag{4.3}$$

Hence when the torque T is applied to the apparatus, it behaves as if it has an additional mass M_c due to its coupling with the superfluid. There would then be a corresponding change in the resonant frequency of the torsional oscillator. The expression (4.3) for this ‘coupling mass’ M_c differs from the expression obtained by Chiao (1982). This is partly because the latter paper assumes that the phase shift is $(m/\hbar)\oint \boldsymbol{v}_a \cdot \mathrm{d}\boldsymbol{r}$ contrary to the treatment above.

The change in the coupling mass due to a small perturbation δv_a of the velocity of rotation of the apparatus is given by (4.3) and (3.4′) to be

$$\delta M_c = \frac{\mathrm{d}M_c}{\mathrm{d}v_a}\delta v_a = \frac{M_s v_0 \sin(v_s/v_q)\delta v_a}{v_q^2[1 + (v_0/v_q)\cos(v_s/v_q)]^3}. \tag{4.4}$$

The sensitivity of detection of the Josephson current due to a perturbation by the measurement of the corresponding change in the coupling mass is therefore proportional to $\mathrm{d}M_c/\mathrm{d}v_a$. Figure 3 illustrates the dependence of $\mathrm{d}M_c/\mathrm{d}v_a$ on v_s for $v_0 < v_q$.

The coupling energy E_c is defined as the part of the energy given to the apparatus to accelerate it from rest to the given angular velocity it now has, due entirely to the coupling of the apparatus to the superfluid. We shall now determine E_c since it will

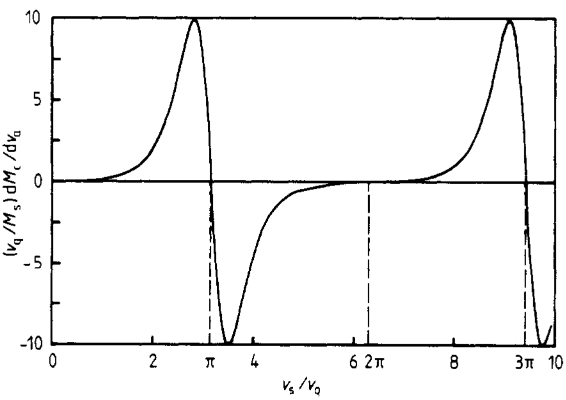

Figure 3. $(v_q/M_s)\,\mathrm{d}M_c/\mathrm{d}v_a$ as a function of the phase difference v_s/v_q across the Josephson junction, where M_c is the coupling mass, for $v_0/v_q = 3/4$. The graph is plotted in the limit $l \ll L$.

be relevant to the analysis of thermal fluctuations in § 7. The change in the coupling energy when the apparatus is turned by a distance $\mathrm{d}r_a$, during which the superfluid velocity changes by $\mathrm{d}v_s$, is $\mathrm{d}E_c = M_s(\mathrm{d}v_s/\mathrm{d}t)\,\mathrm{d}r_a = M_s\,\mathrm{d}v_s\,v_a$. (This can be derived alternatively as $\mathrm{d}E_c = M_c(\mathrm{d}v_a/\mathrm{d}t)\,\mathrm{d}r_a = M_c\,\mathrm{d}v_a\,v_a = M_c v_a[1+(v_0/v_q)\cos(v_s/v_q)]\,\mathrm{d}v_s = M_s v_a\,\mathrm{d}v_s$). Hence the coupling energy is

$$E_c = \int_0^{E_c} \mathrm{d}E_c = \int_0^{v_s} M_s[v_s + v_0 \sin(v_s/v_q)]\,\mathrm{d}v_s.$$

Hence

$$E_c = \tfrac{1}{2}M_s v_s^2 + (\hbar/m)I_0(1-\cos\Delta\phi) \tag{4.5}$$

where $\Delta\phi = -v_s/v_q$ and $I_0 \equiv (m/\hbar)M_s v_0 v_q$ is easily seen to be the critical current in the usual Josephson equation.

The first term in (4.5) would be the kinetic energy of the superfluid, if v_s is constant throughout the superfluid including the junction while the second term, which is periodic in $\Delta\phi$, represents the additional energy due to the change in v_s and α inside the junction. The latter term is analogous to the Josephson coupling energy for superconductors. For small $\Delta\phi$, $E_c \simeq \frac{1}{2}M_s v_q^2(1+v_0/v_q)\Delta\phi^2$. This implies, as expected, that $\Delta\phi = 0$ is a point of stable equilibrium.

5. Phase shift due to the general relativistic Lense–Thirring field

It is well known that, according to general relativity, a rotating body slightly drags inertial frames so that the local inertial frames consequently precess relative to the distant stars (Lense and Thirring 1918). Hence if the superfluid Josephson interferometer, described in § 3, is near such a rotating body, then even if it is non-rotating relative to the distant stars, it would in fact be rotating relative to the local inertial frames; consequently there would be a phase shift due to this purely general relativistic effect.

To determine this phase shift, it would be convenient to rewrite (3.6) in terms of a coordinate system fixed to the apparatus in which the metric coefficients $g_{\mu\nu}$ (signature: $+---$) are such that $\boldsymbol{g}_0 \equiv (g_{01}, g_{02}, g_{03}) = -\boldsymbol{v}_a/c$, c being the velocity of light. Then (3.6) reads

$$\delta\phi = -(mc/\hbar)[1+(v_0/v_q)\cos(v_s/v_q)]^{-1} \oint \delta\boldsymbol{g}_0 \cdot \mathrm{d}\boldsymbol{r} \tag{5.1}$$

where $\delta\boldsymbol{g}_0 = -\delta\boldsymbol{v}_a/c$ is the perturbation. As mentioned earlier, $\delta\boldsymbol{g}_0$ can be non-zero due to the presence of a nearby rotating object. The $\delta\boldsymbol{g}_0$ due to a ball rotating about a diameter is called the Lense–Thirring field in the linear approximation, and is given by (Lense and Thirring 1918)

$$\delta\boldsymbol{g}_0 = (4GMR^2/5c^3r^3)\boldsymbol{\Omega}\times\boldsymbol{r} \tag{5.2}$$

where M, R and $\boldsymbol{\Omega}$ are respectively the mass, radius and angular velocity of the ball, and $\boldsymbol{r}$ is the position vector from the centre of the ball. So if the toroidal tube goes around the great circle of the ball perpendicular to the axis of rotation N times then the phase shift, from (5.1) and (5.2), is (Anandan 1981c)

$$\delta\phi = \frac{8\pi NGMm\Omega R}{5\hbar c^2[1+(v_0/v_q)\cos(v_s/v_q)]} \tag{5.3}$$

where $v_q = \hbar/mL = \hbar/m2\pi RN$.

Similarly the change in the coupling mass due to the Lense–Thirring field of the ball is given by (4.4) to be

$$\delta M_c = \frac{4GM\Omega M_s v_0 \sin(v_s/v_q)}{5c^2 v_q^2[1+(v_0/v_q)\cos(v_s/v_q)]^3}. \tag{5.4}$$

If the apparatus is at rest relative to the distant stars then, of course, $v_s = 0$ and $v_a = 0$ in (5.3) and (5.4). However, if the entire apparatus is attached to the earth then, in general, v_a and v_s would be non-zero because of the earth's rotation.

Consider now the possible detection of the Lense–Thirring field of the earth. In this case also $\delta \boldsymbol{g}_0$ is given by (5.2) with the mass M_e, radius R_e and angular velocity $\boldsymbol{\Omega}_e$ of the earth replacing M, R and $\boldsymbol{\Omega}$. The phase shift, from (5.1), is

$$\delta\phi = \frac{4GM_e mNA}{5\hbar c^2 R_e[1+(v_0/v_q)\cos(v_s/v_q)]}(3\Omega_e \sin\psi\cos\theta - \Omega_{en}) \tag{5.5}$$

where N is the number of turns of the toroidal tube, A is the area enclosed by each turn, ψ is the latitude, θ is the angle between the normal to the area and the vertical and Ω_{en} is the component of the earth's angular velocity normal to the plane of the interferometer. In all these equations v_s can be expressed in terms of v_a on using (3.3′). Equation (5.5) corrects a result in Anandan (1981b).

The above treatment neglects any possible strong gravitational effect due to the variation of the Newtonian potential around the interferometer. A more general treatment that takes this into account is given in the appendix.

6. Comparison with superconductor

The superconducting analogue of the apparatus considered in the previous sections is a superconducting ring, with a Josephson junction, that rotates about the axis of the ring. As in § 3, we shall treat this case also non-relativistically. Figure 1 will now be taken to represent this superconducting ring with the Josephson junction between points 1 and 2. It is well known that, in this case, the current *relative to the junction* is

$$I = I_0 \sin\Delta\phi^* \tag{6.1}$$

where $\Delta\phi^* = \int_1^2 [\nabla\phi' - (q/\hbar)\boldsymbol{A}]\cdot d\boldsymbol{r}$ is the 'gauge invariant phase difference' across the junction, $\boldsymbol{A}$ is the vector potential, ϕ' the phase relative to the junction and q is the charge of the Cooper pair.

Let $\boldsymbol{v}_a$ be the velocity field of this ring relative to the inertial frame K with respect to which it is rotating. We define the superfluid velocity relative to K to be the gauge invariant quantity

$$\boldsymbol{v}_s = (\hbar/m)(\nabla\phi - (q/\hbar)\boldsymbol{A}) \tag{6.2}$$

where ϕ is the phase relative to K and m is the mass. Then

$$\Delta\phi^* = \frac{m}{\hbar}\int_1^2 \boldsymbol{v}_r \cdot d\boldsymbol{r} \tag{6.3}$$

where

$$\boldsymbol{v}_r = \boldsymbol{v}_s - \boldsymbol{v}_a \tag{6.4}$$

is the velocity relative to the ring. Suppose γ is a curve going around the ring through its interior. Then $\oint_\gamma \nabla\phi \cdot d\boldsymbol{r} = 2\pi n$ or

$$\frac{m}{\hbar}\oint_\gamma \boldsymbol{v}_s \cdot d\boldsymbol{r} = 2\pi n + \frac{q}{\hbar}\oint_\gamma \boldsymbol{A} \cdot d\boldsymbol{r} \tag{6.5}$$

where n is an integer. Using (6.4),

$$\frac{m}{\hbar}\oint_{1231} \boldsymbol{v}_a \cdot d\boldsymbol{r} + \frac{m}{\hbar}\oint_{1231} \boldsymbol{v}_r \cdot d\boldsymbol{r} = 2\pi n + \frac{q}{\hbar}\oint_{1231} \boldsymbol{A} \cdot d\boldsymbol{r}. \tag{6.6}$$

From (6.3) and (6.6)

$$\Delta\phi^* = 2\pi n + \frac{q}{\hbar}\oint_{1231} \boldsymbol{A} \cdot d\boldsymbol{r} - \frac{m}{\hbar}\oint_{1231} \boldsymbol{v}_a \cdot d\boldsymbol{r} - \frac{m}{\hbar}\int_{231} \boldsymbol{v}_r \cdot d\boldsymbol{r}, \tag{6.7}$$

where the integrals are over curves in the interior of the ring with the directions and endpoints as indicated.

Now a crucial difference between the present superconducting case and the superfluid helium case is that the current, outside the junction, flows on the surface of the superconductor, i.e.

$$\boldsymbol{v}_r = \boldsymbol{0} \tag{6.8}$$

in the interior of the superconductor, outside the junction. Hence from (6.1), (6.7) and (6.8),

$$I = I_0 \sin\left(\frac{q}{\hbar}\oint_{1231} \boldsymbol{A} \cdot d\boldsymbol{r} - \frac{m}{\hbar}\oint_{1231} \boldsymbol{v}_a \cdot d\boldsymbol{r}\right) \tag{6.9}$$

for this superconducting Josephson interferometer. On the other hand, for the superfluid helium case considered in § 3, (6.8) is not valid, because the superfluid is assumed to flow uniformly, whereas

$$q = 0. \tag{6.10}$$

So in this case (6.1), (6.7) and (6.10) give

$$I = I_0 \sin\left(-\frac{m}{\hbar}\oint_{1231} \boldsymbol{v}_a \cdot d\boldsymbol{r} - \frac{m}{\hbar}\int_{231} \boldsymbol{v}_r \cdot d\boldsymbol{r}\right). \tag{6.11}$$

Also $I = \rho A v_r$, where ρ is the density of the superfluid helium and A is the area of cross section of the tube. Therefore, writing $I_0 = \rho A v_0$, from (6.11),

$$-v_r = v_0 \sin\left(\frac{m}{\hbar}\oint_{1231} \boldsymbol{v}_a \cdot d\boldsymbol{r} + \frac{m}{\hbar}\int_{231} \boldsymbol{v}_r \cdot d\boldsymbol{r}\right)$$

or

$$-v_r = v_0 \sin[m v_a L/\hbar + m v_r (L - l)/\hbar]. \tag{6.12}$$

It is clear, on using (6.4), that this is the same as (3.3).

Hence, the superconducting and the superfluid helium cases can be obtained as special cases, corresponding to (6.8) and (6.10) respectively, of the more general equation (6.7). We also note in this connection that when the approximation $l \ll L$, made previously, is not valid, it appears more convenient to study the relationship between the variables v_r and v_a given by (6.12) rather than v_s and v_a given by (3.3).

Another important difference between the superconducting and the superfluid helium cases is the following. The kinetic momentum of a Cooper pair, according to (6.2), is

$$\boldsymbol{p} \equiv m\boldsymbol{v}_s = \hbar\nabla\phi - q\boldsymbol{A}.$$

Hence, if the magnetic field is turned on then it is possible for $\boldsymbol{A}$ and ϕ to change simultaneously and quickly so that $\boldsymbol{p}$ remains constant (usually $\boldsymbol{0}$). But for superfluid helium, since $q = 0$, changes in phase must necessarily be accompanied by changes in $\boldsymbol{p}$. And changes in the kinetic momentum $\boldsymbol{p}$ propagate with the velocity of sound. Hence, while it is known that the time taken for the phase difference to accumulate across the Josephson junction in a superconducting ring is $\hbar/\Delta$, where Δ is the energy gap, the corresponding time for the superfluid helium interferometer cannot be less than L/v, where v is the velocity of sound in superfluid helium and L is the length of the tube.

Hence if the superfluid helium interferometer is used to detect gravitational radiation (Chiao 1981, 1982, Anandan 1981b) then the speed of propagation of the signal inside the superfluid cannot exceed the velocity of sound v. This disagrees with the conclusion of Chiao (1981, 1982) that this speed is the velocity of light. Thus while a SQUID can be used to detect electromagnetic waves, because of the quickness with which the phase builds up across the Josephson junction for reasons mentioned above, the superfluid helium gravitational wave detector, which is no more efficient than a Weber bar, is probably not suitable to detect gravitational waves. But this objection does not apply to possible detection of the Lense–Thirring field because the signal, in this case, is time independent, unlike in the case of gravitational radiation.

7. Thermal fluctuations

We now investigate the limitations in sensitivity of the apparatus considered in §§ 3–5 due to thermal fluctuations of the superfluid. The general theory of thermal fluctuations from thermodynamic equilibrium will first be outlined and then the special case of the superfluid Josephson interferometer will be considered.

Consider a system σ in thermal equilibrium with a thermal bath so that the combined system of σ and the bath is a closed system. It is known that the probability of fluctuation from equilibrium $w \propto \exp(\Delta S_t)$ where ΔS_t is the change in the total entropy of the combined system. It can be shown that (Landau and Lifshitz 1969)

$$\Delta S_t = -R_{\min}/kT \tag{7.1}$$

where $R_{\min}$ is the minimum work on σ needed to carry out the given change in the thermodynamic quantity X of σ whose fluctuation is of interest, T is the absolute temperature and k is the Boltzmann constant. Suppose $E(X)$ is the internal energy of σ. Then if σ does not undergo a change in volume and no part of $R_{\min}$ is given away as heat to the thermal bath then

$$R_{\min} = \Delta E = \tfrac{1}{2}\,\partial^2 E/\partial X^2|_{X_0}(\Delta X)^2 + \mathrm{O}[(\Delta X)^3] \tag{7.2}$$

where $X = X_0$ corresponds to stable equilibrium so that $\partial E/\partial X|_{X_0} = 0$ and $\partial^2 E/\partial X^2|_{X_0} > 0$. It therefore follows from (7.1) and (7.2) that

$$w \propto \exp[-\tfrac{1}{2}\partial^2 E/\partial X^2|_{X_0}(\Delta X)^2/kT]. \tag{7.3}$$

Hence

$$\overline{(\Delta X)^2} = \frac{kT}{\partial^2 E/\partial X^2|_{X_0}} \tag{7.4}$$

where the bar denotes thermal average. It is therefore necessary to compute $\partial^2 E/\partial X^2|_{X_0}$ to determine the mean square fluctuations of X.

Consider now the superfluid Josephson interferometer, described earlier, which has constant angular velocity along its axis relative to an inertial frame K and is in thermal equilibrium with a thermal bath. Let F be a rotating frame such that the apparatus is at rest with respect to F when it is in thermal equilibrium with the bath. If the apparatus is at rest on the earth, F would be the frame of the earth. The superfluid is taken to be the system σ in the above analysis. Let v_s be the superfluid velocity and v_a the velocity of the apparatus relative to K. At equilibrium, let $v_s = v_{s0}$ and $v_a = v_{a0}$. Then $v_s' = v_s - v_{a0}$ and $v_a' = v_a - v_{a0}$ are the superfluid velocity and apparatus velocity, relative to F.

Suppose E_c' is the internal energy of the superfluid relative to F. Then the change in E_c' when the apparatus is rotated infinitesimally is

$$\mathrm{d}E_c' = M_s\,\mathrm{d}v_s'\,v_a' = M_s\,\mathrm{d}v_s(v_a - v_{a0}) = M_s\,\mathrm{d}v_s[v_s + v_0\sin(v_s/v_q) - v_{a0}]$$

on using (3.3′). Hence

$$E_c' = \int_{v_{s0}}^{v_s} \mathrm{d}E_c' = \tfrac{1}{2}M_s(v_s^2 - v_{s0}^2) + (\hbar/m)I_0(\cos\Delta\phi_0 - \cos\Delta\phi) + M_s v_{a0}(v_{s0} - v_s) \tag{7.5}$$

where $\Delta\phi = v_s/v_q$, $\Delta\phi_0 = v_{s0}/v_q$ and $I_0 = mM_s v_0 v_q/\hbar$. Clearly (4.5) is a special case of (7.5) corresponding to $v_{s0} = 0$ and $v_{a0} = 0$. Now $\partial E_c'/\partial v_s|_{v_s = v_{s0}} = 0$. Therefore if $\delta\phi_T$ is the thermal fluctuation in the phase shift, then using (7.4)

$$(\delta\phi_T)^2 \equiv \frac{\overline{(\delta v_s)^2}}{v_q^2} = \frac{kT}{v_q^2\,\partial^2 E/\partial v_s^2|_{v_{s0}}} = \frac{kT}{M_s v_q^2[1 + (v_0/v_q)\cos\Delta\phi_0]}. \tag{7.6}$$

Since $\delta v_a = (\mathrm{d}v_a/\mathrm{d}v_s)\delta v_s = [1 + (v_0/v_q)\cos\Delta\phi_0]v_q\delta\phi$, the smallest δv_a that can be measured from the corresponding phase shift $\delta\phi$, allowed by the thermal fluctuation (7.6), is

$$(\delta v_a)_T = [1 + (v_0/v_q)\cos\Delta\phi_0]^{1/2}(kT/M_s)^{1/2}. \tag{7.7}$$

The coupling mass relative to F is given by

$$M_c' = \frac{1}{v_a'}\frac{\mathrm{d}E_c'}{\mathrm{d}v_a'} = \frac{1}{v_a - v_{a0}}\frac{\mathrm{d}E_c'}{\mathrm{d}v_s}\frac{\mathrm{d}v_s}{\mathrm{d}v_a} = \frac{M_s}{1 + (v_0/v_q)\cos\Delta\phi} = M_c \tag{7.8}$$

on using (3.3′). Hence the change in the coupling mass when v_a changes by δv_a is

$$\delta M_c = \frac{\mathrm{d}M_c'}{\mathrm{d}v_s}\frac{\mathrm{d}v_s}{\mathrm{d}v_a}\delta v_a = -\frac{M_s(v_0/v_q^2)\sin\Delta\phi_0\,\delta v_a}{[1 + (v_0/v_q)\cos\Delta\phi_0)]^3}. \tag{7.9}$$

Consider now the special case of $\cos\Delta\phi_0 = 0$. Then, from (7.7), $(\delta v_a)_T = (kT/M_s)^{1/2}$. The δv_a due to the earth's Lense–Thirring fields is $(\delta v_a)_e = R\delta\Omega_e$, where R is the radius of the torus and $\delta\boldsymbol{\Omega}_e$ is the Lense–Thirring precession of inertial frames due to the earth's rotation near the surface of the earth where the experiment is being performed. From (5.2), for the earth's Lense–Thirring field,

$$\delta\boldsymbol{\Omega}_e = \tfrac{1}{2}c \text{ curl } \delta\boldsymbol{g}_0 = \frac{4GM_eR_e^2}{5c^2}\left(-\frac{\boldsymbol{\Omega}}{r^3} + \frac{3(\boldsymbol{\Omega}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5}\right). \tag{7.10}$$

Therefore $\delta\Omega = 4.5\times10^{-14}$ rad s^{-1} on the earth' surface ($r = R_e$) at latitude 45°. So if $R = 3m$ then $(\delta v_a)_e = 1.35\times10^{-13}$ m s^{-1}. Now the temperature and the mass of the superfluid needed, so that the corresponding phase shift is bigger than the thermal fluctuation, must satisfy $(\delta v_a)_e \geqslant (kT/M_s)^{1/2}$. For $T = 1$ K, therefore $M_s \geqslant 7.5\times10^2$ kg. This minimum mass needed can, of course, be decreased by lowering the temperature and/or increasing the radius of the torus.

8. Conclusion

We have developed a general theory of the Josephson effect in superfluid helium which gives definite experimental predictions for superfluid helium in a toroidal tube with a Josephson junction. An interesting outcome of this analysis is that when $v_0 > v_q$, hysteresis can occur as seen from figure 2(*b*). In this case, for a given value of the apparatus velocity v_a, several values of the superfluid velocity v_s and therefore the Josephson current is possible and transitions between these various values may take place. So the experimental difficulty in detecting this effect may be due to the experiments, so far, being done for $v_0 > v_q$.

Also, as seen from figure 3, the sensitivity of detection of this effect is minimum when v_s and therefore v_a is zero. So it would be necessary for the apparatus to have already an angular velocity such that $|dM_c/dv_a|$ has a large value. Hence if the experiment is redone under these conditions, it may be possible to detect this effect. Once this effect has been unambiguously detected, we would be in a much better position to assess the possibility of detection of the phase shift due to the general relativistic Lense–Thirring field, which is predicted in § 5.

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Appendix. The superfluid Josephson interferometer in a stationary gravitational field

The phase shift in a Josephson interferometer which would be valid even in a strong gravitational field was obtained by Anandan (1981b) for the special case when the velocity of the superfluid, relative to the apparatus outside the Josephson junction, is negligible. While this assumption is valid for a superconducting interferometer, as already mentioned in § 6, since the Cooper pairs are pinned to the lattice in the interior of a superconductor; it is in general not a good approximation for superfluid helium. We consider here the same problem for superfluid helium in the more general case when this relative motion is not negligible, which was treated for weak gravitational field by Anandan (1981c).

The superfluid 'four-velocity' v^μ, which is defined by $v_\mu = -(\hbar/mc)\,\partial_\mu \tilde{\phi}$, where the scalar $\tilde{\phi}$ is the phase of the relativistic order parameter $\tilde{\Psi}(\boldsymbol{r}, t)$, satisfies

$$\nabla_\mu v_\nu - \nabla_\nu v_\mu = 0 \tag{A1}$$

and

$$v^\mu v_\mu = 1 + f \tag{A2}$$

where f is a scalar function defined in Anandan (1981a, b) and ∇_μ is the covariant derivative. This v^μ also satisfies the continuity equation (Anandan 1981a, b): $\nabla_\mu(\alpha^2 v^\mu) = 0$. The phase difference across the Josephson junction that enters into the Josephson equation $I = I_0 \sin \Delta\tilde{\phi}$ is then

$$\Delta\tilde{\phi} = -\frac{mc}{\hbar}\int_\gamma v_\mu \,\mathrm{d}x^\mu, \tag{A3}$$

where γ is a curve going around the tube, through the superfluid, beginning and ending at two events that are simultaneous relative to the Josephson junction on the opposite sides of the junction. We assume that the gravitational field has a Killing field ξ^μ such that $v^\mu = \rho\xi^\mu$, outside the junction, where the function ρ will now be determined.

Let s^μ be a vector field such that $\xi^\mu s_\mu = 0$, $s^\mu s_\mu = -1$ and s^μ is normal to each cross section of the tube containing the superfluid helium. It then follows from (A1) that $0 = s^\mu \xi^\nu(\nabla_\mu v_\nu - \nabla_\nu v_\mu) = \lambda s^\mu\,\partial_\mu\rho + \rho s^\mu \xi^\nu\,\nabla_\mu \xi_\nu - \rho s^\mu \xi^\nu\,\nabla_\nu \xi_\mu$ where $\lambda \equiv \xi^\mu \xi_\mu$. On using Killing's equation $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$, therefore $\rho s^\mu\,\partial_\mu \lambda = -\lambda s^\mu\,\partial_\mu \rho$. Hence, $\rho = A\lambda^{-1}$ with $s^\mu\,\partial_\mu A = 0$. We assume a stationary situation for which $\xi^\mu\,\partial_\mu A = 0$. Then A is a constant, which can be chosen to be 1 by appropriately normalising ξ^μ. Hence, from (A3),

$$\Delta\tilde{\phi} = -\frac{mc}{\hbar}\int_\gamma \lambda^{-1}\xi_\mu \,\mathrm{d}x^\mu \tag{A4}$$

which is the same as equation (11) of Anandan (1981b), (when the size of the junction is negligible).

Choose now a coordinate system comoving with the superfluid in which $\xi^\mu = (1, 0, 0, 0)$. Then $\lambda^{-1}\xi_\mu = g_{00}^{-1} g_{\mu 0} \simeq (1, g_{01}, g_{02}, g_{03})$ for a weak gravitational field, neglecting terms that are second order in $(g_{00} - 1)$, $g_{0i} (i = 1, 2, 3)$. Then (A4) is

$$\Delta\tilde{\phi} = -\frac{mc}{\hbar}\int_\gamma g_{0i} \,\mathrm{d}x^i, \tag{A5}$$

which is the same as (3.3) of Anandan (1981c). The g_{0i}'s in (3.3) and (3.4) of the latter paper, which are defined in two different coordinate systems stated above these equations, have the values $-v_s^i/c$ and $-v_a^i/c$, in the special case when the apparatus is at rest with respect to the second coordinate system. The relation between these two g_{0i}'s are then given by (3.3′) of the present paper, which is also an immediate consequence of (3.1), (3.2) of Anandan (1981c) (for a weak gravitational field). The former g_{0i} is determined by solving this equation and (A5) is then evaluated. This was done in the latter paper for the more general case when the apparatus is not necessarily at rest in the second coordinate system.

Suppose now that there is a Killing field ξ'^{μ} whose direction is the same as that of t^{μ}, the four-velocity field of the apparatus, and not that of v^{μ}. Then $v^{\mu}\nabla_{\mu}(\xi'^{\nu}v_{\nu}) = v^{\mu}\xi'^{\nu}\nabla_{\mu}v_{\nu} = \frac{1}{2}\xi'^{\nu}\partial_{\nu}f$ where we have used (A1), (A2) and the Killing equation for ξ'_{ν}. In the present stationary situation we can assume that $\xi'^{\nu}\partial_{\nu}f = 0$ and $\xi'^{\nu}\partial_{\nu}(\xi'^{\mu}v_{\mu}) = 0$. Then $\xi'^{\nu}v_{\nu}$ is a constant ω_0, say. Therefore we can write $v^{\mu} = (\lambda')^{-1}\omega_0\xi'^{\mu} + \kappa s'^{\mu}$ where $\lambda' = \xi'^{\mu}\xi'_{\mu}$ and s'^{μ} is orthogonal to ξ'^{μ} and the cross section of the tube and $s'^{\mu}s'_{\mu} = -1$. From (A2), κ and ω_0 are related by $(\lambda')^{-1}\omega_0^2 - \kappa^2 = 1 + f$. Also, from (A3),

$$\Delta\tilde{\phi} = -\frac{mc}{\hbar}\int_{\gamma}((\lambda')^{-1}\omega_0\xi'_{\mu} + \kappa s'_{\mu})\,\mathrm{d}x^{\mu}. \tag{A6}$$

which can now be evaluated on using the continuity equation, stated below (A2), and the Josephson equation.

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